SOLUTION OF PARTIAL DIFFERENTIAL EQUATIONS LIEBMANN METHOD

by

LARRY WAYNE OST

B.S., Jamestown College, 1964

A MASTER'S REPORT

submitted in partial fulfillment of the

requirements for the degree

MASTER OF SCIENCE

Department of Mathematics

KANSAS STATE UNIVERSITY
Manhattan, Kansas

1968

Approved by:

Leonard R. Fuller

Major Professor

TABLE OF CONTENTS

INTRODUCTION

In the theory of linear differential equations of the elliptic type, boundary-value problems play a basic role. Such problems involve the determination of the solution of a given differential equation within a given domain on the boundary of which given conditions must be fulfilled.

Partial differential equations in two variables involve a domain of application in a two-dimensional plane. Boundary values may be assigned in various ways along curves in the plane to determine the solution elsewhere in the plane. For second-order equations, a classification into three basic types is useful in distinguishing boundary value problems.

Consider a second order partial differential equation of the form

$$[1] \quad a(x,y)\phi_{xx} + b(x,y)\phi_{xy} + c(x,y)\phi_{yy} + f(x,y,\phi_x,\phi_y) = 0,$$

i.e., one that is linear in the second-order partial derivatives. It is possible to classify this equation on the basis of the expression $b^2 - 4ac$, where a, b and c are the coefficients of the second order terms of the equation. The classification is as follows:

1. Parabolic Type: (if $b^2 - 4ac = 0$ at all points of the region considered). An example of this type is the one-dimensional heat-flow equation.

2. Hyperbolic Type: (if $b^2 - 4ac > 0$ for all points of the region). A basic problem of this type is the one-dimensional

wave equation.

3. Elliptic Type: (if $b^2 - 4ac < 0$ at all points of the region). The most fundamental equation of this type is Laplace's equation. The elliptic equation will be considered in this report. If, in the elliptic type equation, b and f are both zero and a = c = 1, the equation is called Laplace's equation in two dimensions and its solution is referred to as a harmonic function.[1]

Any elliptic equation with f = 0 can be reduced to Laplace's equation. First of all, a rotation of axes can be used to remove the mixed partial term. A common transformation to accomplish this is

$$x = X\cos\phi - Y\sin\phi, \qquad y = X\sin\phi + Y\cos\phi$$

where ϕ is the angle of rotation.[2]

The general equation then reduces to the form

$$a'(X,Y)\phi_{XX} + c'(X,Y)\phi_{YY} = 0.$$

Then a change of scale can be used to make the coefficients of the second partials equal.

Boundary-value problems of this type involve specifying values of the function or of its partial derivatives of first

---

1. A solution of Laplace's equation (in any number of dimensions) is called a harmonic function.

2. Derived in Calculus, Tom. M. Apostol, Blaisdell Publishing Company, New York, London, 1961, pp. 318-320.

order along the closed boundary of the region. Then solutions are determined for all interior points of the region. A typical boundary value problem for elliptic equations is the Dirichlet problem which may be defined as follows:

DEFINITION: (Dirichlet Problem): "Let f be a continuous function prescribed on the boundary B of some finite region G of the space $(x_1, x_2, \ldots, x_n)$. We are to find a function $\phi(x_1, x_2, \ldots, x_n)$ which is harmonic in the interior of G and equal to f on B."

In general, numerical methods of solution of all Dirichlet boundary-value problems depend on the use of a grid of points in the domain. At the nodes of this grid, approximations to the solution are determined. With an infinitely fine mesh, the solution would converge to the solution of the partial differential. Various coordinate meshes may be used in the two dimensional system; however, the configuration in this report is restricted to a square lattice.

Laplace's equation in two dimensions,

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0,$$

is sometimes written $\nabla^2 \phi = 0$. The expression $\nabla^2 \phi$, whether referring to a rectangular coordinate system as above, in two or more dimensions, or the corresponding expression in some other system of coordinates, is called the Laplacian.

In a square lattice the two-dimensional Laplacian can be approximated by replacing the second derivatives by their

expressions in terms of finite differences. It is customary to use only the approximation to the Laplacian obtained by neglecting the fourth and higher differences. If a single subscript is used to designate the relative position of the ϕ's in the lattice, the expression to evaluate the potential function ϕ at each interior point looks simpler, compared with using double subscripts. However, a system must be defined for a consistent numbering pattern of the lattice points. Then the Laplacian, expressed in terms of finite differences, may be written as[3]

$$\nabla_h^2 \phi_0 = 1/h^2 \, (\phi_1 + \phi_2 + \phi_3 + \phi_4 - 4 \, \phi_0),$$

where ϕ_1, ϕ_2, ϕ_3, and ϕ_4 are the mesh points h units from ϕ_0. In symbolic notation

$$\nabla_h^2 \phi_0 = 1/h^2 \begin{Bmatrix} & \phi_2 & \\ \phi_3 & -4 \, \phi_0 & \phi_1 \\ & \phi_4 & \end{Bmatrix}$$

This report is based on the Liebmann method of solving Laplace's equation. In general, the Liebmann method is a process for evaluating the potential function at an interior point in terms of its neighboring points. The method and its extensions are illustrated in the following sections.

---

3. Kaiser S. Kunz, Numerical Analysis, McGraw-Hill Book Company, Inc., New York, London, Toronto, 1957, pp. 278-79.

LIEBMANN METHOD - TWO DIMENSIONAL

The method of Liebmann can best be illustrated by a simple example, derived with the use of Laplace's equation. Assume that the potential ϕ, satisfying Laplace's equation, is required for the region contained within a given square boundary (see figure 1). The potential ϕ is zero on three sides of the square. On the fourth side $\phi = 1$ at the middle and drops off linearly to zero at the corners.

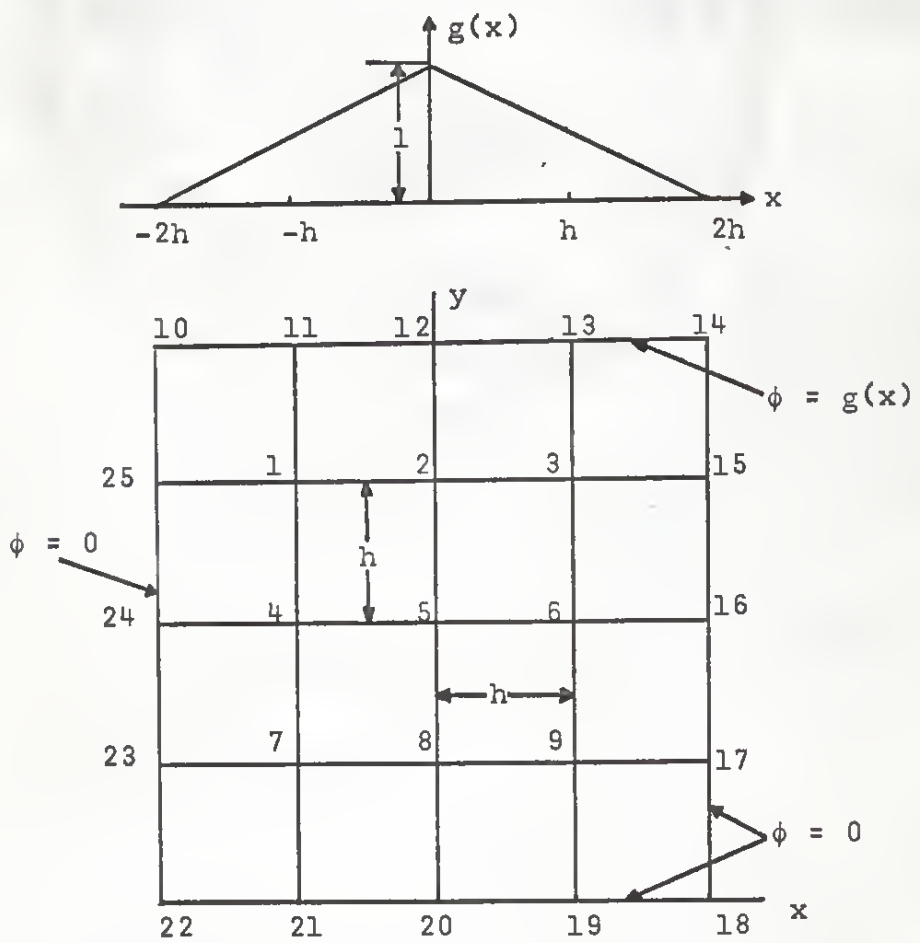


Fig. 1

In this example, a square net of 25 lattice points is used to approximate the behavior of ϕ. The potential at the interior lattice points can be designated by ϕ_s where s = 1, 2, ... , 9; the potential at the boundary points by ϕ_t, t = 10, 11, ... , 25. The problem then is to determine a numerical solution of the ϕ_s in terms of the ϕ_t.

It has already been shown that Laplace's equation can be approximated by the partial difference equation indicated symbolically by

$$[2] \qquad \nabla_h^2 \phi_s = 1/h^2 \begin{Bmatrix} & 1 & \\ 1 & -4 & 1 \\ & 1 & \end{Bmatrix} \phi_s = 0,$$

where h is the net spacing. Consider the potential at interior point ϕ_1. The difference equation which expresses the potential for this point is written

$$1/h^2 \, (\phi_{11} + \phi_{25} + \phi_2 + \phi_4 - 4 \, \phi_1) = 0.$$

We can solve this equation for $4 \, \phi_1$ and write it as

$$4 \, \phi_1 = \phi_{11} + \phi_{25} + \phi_2 + \phi_4 \, ,$$

or write it in the following form which allows only the boundary points on the right hand side of the equation:

$$[3] \qquad -4 \, \phi_1 + \phi_2 + \phi_4 = - \, (\phi_{11} + \phi_{25})$$

Equation 3 gives the potential at the first interior point $\phi_s = 1$. As we consider the remaining interior points,

$s = 2, 3, \ldots, 9$, we have eight additional equations.

$$\phi_1 - 4\phi_2 + \phi_3 + \phi_5 = -\phi_{12}$$

$$\phi_2 - 4\phi_3 + \phi_6 = -(\phi_{13} + \phi_{15})$$

$$\phi_1 - 4\phi_4 + \phi_5 + \phi_7 = -\phi_{24}$$

$$\phi_2 + \phi_4 - 4\phi_5 + \phi_6 + \phi_8 = 0$$

$$\phi_3 + \phi_5 - 4\phi_6 + \phi_9 = -\phi_{16}$$

$$\phi_4 - 4\phi_7 + \phi_8 = -(\phi_{21} + \phi_{23})$$

$$\phi_5 + \phi_7 - 4\phi_8 + \phi_9 = -\phi_{20}$$

$$\phi_6 + \phi_8 - 4\phi_9 = -(\phi_{17} + \phi_{19})$$

Notice that the right hand sides of these equations are known as they represent values at the boundary points which are constant. We have a system of nine equations in as many unknowns which can be solved by standard methods. However, if the system should be expanded and a large number of interior points considered, the method would become an exercise in "busy work". For this reason we look to other methods.

Consider the set of linear equations: equation 3 and the eight additional equations for this problem. The matrix of the coefficients for this system of equations has -4 as the elements of its principal diagonal. Notice that all of the other elements are either 0 or 1. Hence, the diagonal elements are dominant. In fact, the absolute value of any element on the diagonal is at least as large as the sum of the remaining elements in that row or column. The nature of this coefficient matrix indicates that

we can safely look to some method of successive approximations for our solution.[4] One such method is introduced by Liebmann.[5]

If the symbolic difference equation expressed above is rewritten, we have, symbolically,

$$[4] \qquad \phi_s = 1/4 \left\{ \begin{array}{ccc} & 1 & \\ 1 & 0 & 1 \\ & 1 & \end{array} \right\} \phi_s .$$

This equation states that ϕ_s is just an average of the values of ϕ at the four neighboring points. Neighboring points are defined as those points a distance h away from ϕ. Liebmann's method consists of improving the value initially guessed for ϕ by repeated application of this process over the set of points.

One passes from point to point in the lattice replacing the previous values of ϕ at each point by the average of the ϕ's for the four closest neighboring points. All points in the interior of the region must be included in the process, although it does not matter what sort of a pattern is originally set up for each region.

As illustrated in figure 1, one possible pattern for numbering the lattice points is to begin at the upper left interior point. Number consecutively along the rows, not including any point falling on the boundary of the lattice. The

---

4. Ibid., pg. 296.

5. H. Liebmann, Die Angenaherte Ermittlung Harmonischer Funktionen und Konformer Abbildungen, Sitzber, Math. Physik, kl. Boyer. Alcad. Wiss. Munchen, 47: 385-416 (1918).

potential is evaluated for ϕ_1. Then the potential is evaluated for the interior point of next higher index. The process continues in the defined order throughout the entire interior of the lattice. When the potential at all points has been improved once, these new values are substituted for the values of the previous iteration and the process begins again at ϕ_1.

The process converges since each application of the method averages the errors of the four neighboring points. If the errors are not all of the same sign, they will tend to average out and ϕ_s will be improved. In several cases, one or more of the four neighboring points is actually a boundary point which has no error. Although the process converges, in some problems the convergence is rather slow.

The Liebmann process is well suited to a large scale computer: however, the requirement that values for all ϕ's in the lattice must be made available over and over again places restrictions on the number of lattice points used. With today's bulk storage computers this problem is reduced to a minimum.

One other preliminary must be disposed of before a problem of this type can be read into a computer for calculation. We must have a starting point. An initial guess must be made for the values of ϕ at the interior points. A typical boundary-value problem for elliptic equations is stated in the definition of the Dirichlet problem on page 3. Using this definition, we will prove the following theorem:

THE MINIMUM-MAXIMUM THEOREM: Consider a harmonic function $\phi(x,y)$, continuous in some closed bounded region $\overline{G} = G + B$. Then the

values of ϕ in G cannot exceed its maximum on the boundary B nor can they be less than its minimum on B.

PROOF: Let m denote the maximum of ϕ on B. Assume that, at some points of G, ϕ assumes values greater than m. Then the maximum M of ϕ in $\overline{G}$ must also be greater than m, and this maximum must be assumed at some interior point Q of G. We now translate the origin to the point Q. Under this transformation ϕ remains harmonic. Now consider the function

$$v(x,y) = \phi(x,y) + \frac{M - m}{2d^2}(x^2 + y^2),$$

where d is equal to the least upper bound of the distances between pairs of points of B (the maximum distance across the region). If (x,y) is in G, then $x^2 + y^2 < d^2$. From the above equation, we can see that $v(0,0) = \phi(0,0)$ which is equal to M. On the other hand, if a point (x,y) belongs to the boundary B of $\overline{G}$, then

$$v(x,y) \leq m + (1/2)(M - m) = (1/2)(m + M) < M.$$

This can be obtained by noting that $\phi(x,y) \leq m$ and since $x^2 + y^2 < d^2$, it follows that

$$\frac{x^2 + y^2}{d^2} < 1.$$

Consequently, v(x,y), like $\phi(x,y)$, must attain its maximum at an interior point of G. However, for all points of G, we have

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{2(M - m)}{d^2} > 0.$$

This contradicts the fact that for a maximum, none of the second derivatives of a function can be positive.

To prove the minimum part of the theorem, apply the above result to $-\phi(x,y)$. This means that the maximum value of $-\phi(x,y)$ is the minimum value of $\phi(x,y)$ and hence, the minimum cannot occur at an interior point.

The initial values assigned to ϕ could be zero. However, the theorem indicates that the guesses should lie somewhere between the minimum and maximum observed on the boundary. When we intend to find a solution for such a problem using a computer, the accuracy of the guess is not of major significance. At worst, a bad guess would only require more iterations. After approximating the potential at all interior points once (one complete iteration), we have a set of values as shown in table 1, appendix A. The final solution for this example, along with a FORTRAN program, is also illustrated in appendix A.

The approximation to the Laplacian, as defined above, is derived by neglecting fourth and higher differences in a difference equation.[6] If this same difference equation is considered, neglecting only sixth and higher differences, we have another approximation to the Laplacian. Thus, the Laplacian may

---

6. This discussion is illustrated in detail in Numerical Analysis by K. S. Kunz, pp. 279-80.

be expressed as

$$\nabla_h^2\phi_{00} = \frac{1}{h^2}\left(\delta_x^2\phi_{00} - \frac{1}{12}\,\delta_x^4\phi_{00} + \delta_y^2\phi_{00} - \frac{1}{12}\,\delta_y^4\phi_{00}\right)$$

$$= \frac{1}{12h^2}\left[-(\phi_{20} + \phi_{02} + \phi_{-20} + \phi_{0-2}) + 16(\phi_{10} + \phi_{01} + \phi_{-10} + \phi_{0-1}) - 60\,\phi_{00}\right],$$

or, symbolically,

$$\nabla_h^2\phi_{00} = \frac{1}{12h^2}\begin{Bmatrix} & & -1 & & \\ & & 16 & & \\ -1 & 16 & -60 & 16 & -1 \\ & & 16 & & \\ & & -1 & & \end{Bmatrix}$$

This approximation should yield better results than the Laplacian derived for four neighboring points. It is very seldom used, however, due to the added complications. We now must know the potential at points adjacent to, but outside the boundary or we must assign weights other than those illustrated in the equation above. In figure 1, for example, we do not have two boundary points defined to the left of ϕ_1.

LIEBMANN METHOD - THREE DIMENSIONAL

It would be suitable to consider next the potential when the system is extended to three coordinates. It turns out that for a simple three-dimensional region it is not always possible to solve the Dirichlet problem. A restriction, as stated by I. G. Petrovsky, is that the function have a sufficient number of continuous derivatives on the bounding surface. This condition can be stated that the function at the bounding surface be sufficiently smooth.

The Laplacian of ϕ expressed in Cartesian coordinates in three dimensions is written as

$$\nabla_h^2 \phi_0 = \frac{\partial^2 \phi}{\partial x^2} \quad \frac{\partial^2 \phi}{\partial y^2} \cdot \frac{\partial^2 \phi}{\partial z^2} \quad .$$

Therefore we can use an approximation similar to that used for

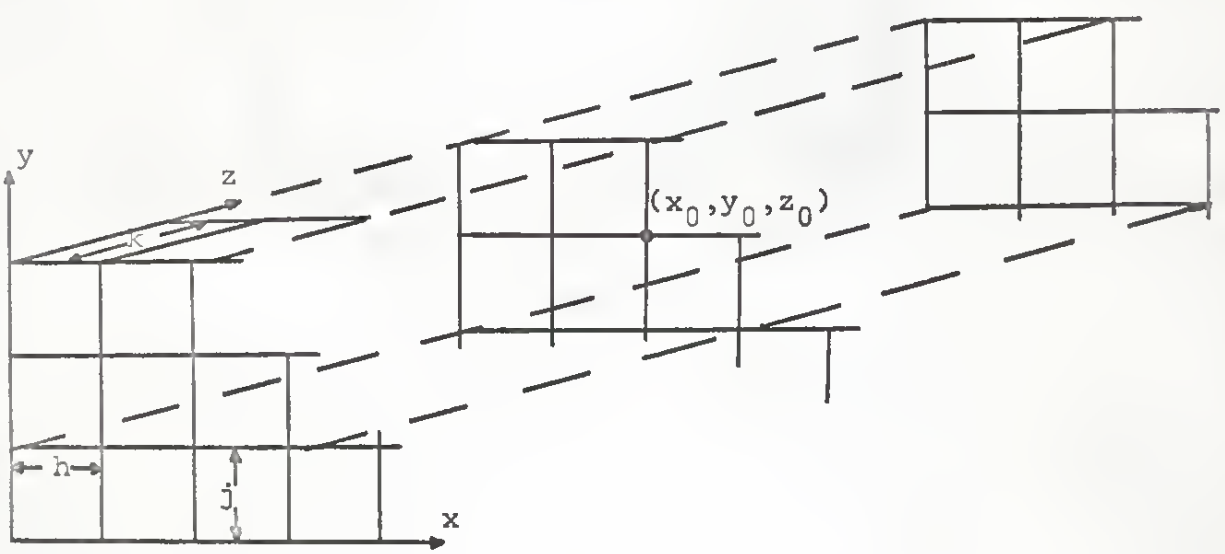


Fig. 2

the two dimensional case. In a three dimensional lattice this Laplacian can be approximated by replacing the second derivatives by their expressions in terms of finite differences.

The Liebmann derivation of a formula for two dimensions is discussed in detail in chapter 12 of Numerical Analysis by K. S. Kunz. This is known as the Laplacian for two dimensions and is also mentioned briefly in the introduction of this report. The following discussion is analogous to that of Kunz. The only significant difference is that we now have a three dimensional array.

Suppose the Laplacian is desired at (x_0,y_0,z_0) in figure 2. We can define u, v and w by the following equations:

$$u = \frac{x - x_0}{h} \qquad v = \frac{y - y_0}{j} \qquad w = \frac{z - z_0}{k}$$

Let

$$\phi_{rst} = \phi(x_r,y_s,z_t) \ .$$

Then by the use of Stirling's interpolation formula.[7]

$$\phi(x,y_0,z_0) = \phi_{000} + u\mu\delta_x\phi_{000} + (1/2!)u^2\delta_x^2\phi_{000} +$$

$$(1/3!)u(u^2 - 1)\mu\delta_x^3\phi_{000} + (1/4!)u^2(u^2 - 1)\delta_x^4\phi_{000} + ---$$

and hence,

---

7. Kaiser S. Kunz, Numerical Analysis, McGraw-Hill Book Company, Inc., New, London, Toronto, 1957, pg. 71.

$$\left.\frac{\partial^2\phi(x,y_0,z_0)}{\partial x^2}\right|_{x=x_0} = \frac{1}{h^2}\left.\frac{\partial^2\phi(x,y_0,z_0)}{\partial u^2}\right|_{u=0}$$

$$= \frac{1}{h^2}\left(\delta_x^2\phi_{000} - \frac{1}{12}\,\delta_x^4\phi_{000} + \text{---}\right)$$

where the subscript x on the δ_x indicates that the differences are formed with respect to x.

$$\delta_x^2\phi_{000} = \phi_{-100} - 2\phi_{000} + \phi_{100}$$

and

$$\delta_x^4\phi_{000} = \phi_{-200} - 4\phi_{-100} + 6\phi_{000} - 4\phi_{100} + \phi_{200} \quad .$$

Since the second derivatives with respect to y and z can be handled similarly, the following equation can be obtained.

$$\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}\right)_{x_0,y_0,z_0} = \frac{1}{h^2}\left(\left(\delta_x^2\phi_{000} - \frac{1}{12}\,\delta_x^4\phi_{000} + \text{---}\right)\right) +$$

[5]

$$\frac{1}{j^2}\left(\left(\delta_y^2\phi_{000} - \frac{1}{12}\,\delta_y^4\phi_{000} + \text{---}\right)\right) + \frac{1}{k^2}\left(\left(\delta_z^2\phi_{000} - \frac{1}{12}\,\delta_z^4\phi_{000} + \text{---}\right)\right)$$

It is customary to use only the approximation to the Laplacian obtained by neglecting the fourth and higher differences in Eq. 5. The results in other cases would be similar. If the lattice is set up so that all net spacings are the same, h = j = k. Therefore

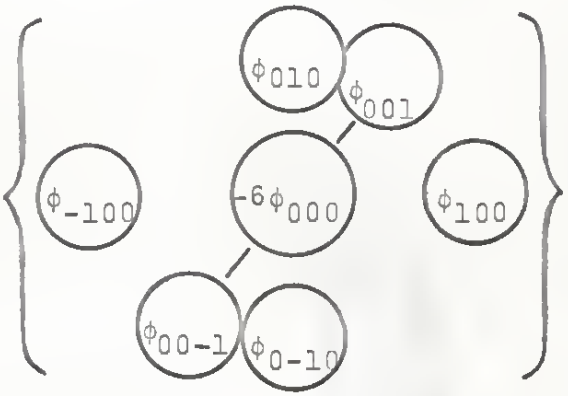
$$\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}\right)_{x_0,y_0,z_0} \cong \nabla_h^2\phi_{000}$$

where $$\nabla_h^2\phi_{000} = \frac{1}{h^2}\left(\delta_x^2\phi_{000} + \delta_y^2\phi_{000} + \delta_z^2\phi_{000}\right)$$

$$= \frac{1}{h^2}\left(\phi_{100} + \phi_{010} + \phi_{001} + \phi_{-100} + \phi_{0-10} + \phi_{00-1} - 6\phi_{000}\right).$$

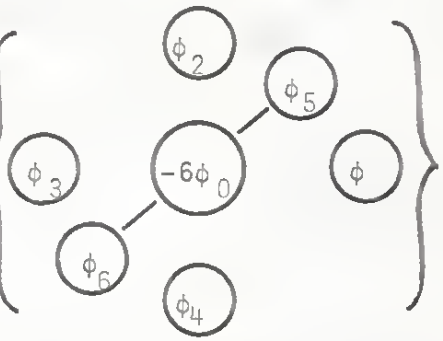
Then, in symbolic notation,

$$\nabla_h^2\phi_{000} = \frac{1}{h^2}$$



Using single subscripts to designate the relative position of the ϕ's in the net, the expression can be written as

$$\nabla_h^2\phi_0 = \frac{1}{h^2}$$



If one should choose to neglect only the sixth and higher differences,

$$\frac{-1}{12h^2}\,\delta_x^4\phi_{000}, \qquad \frac{-1}{12h^2}\,\delta_y^4\phi_{000}, \qquad \frac{-1}{12h^2}\,\delta_z^4\phi_{000}$$

would be evaluated.

Hence,

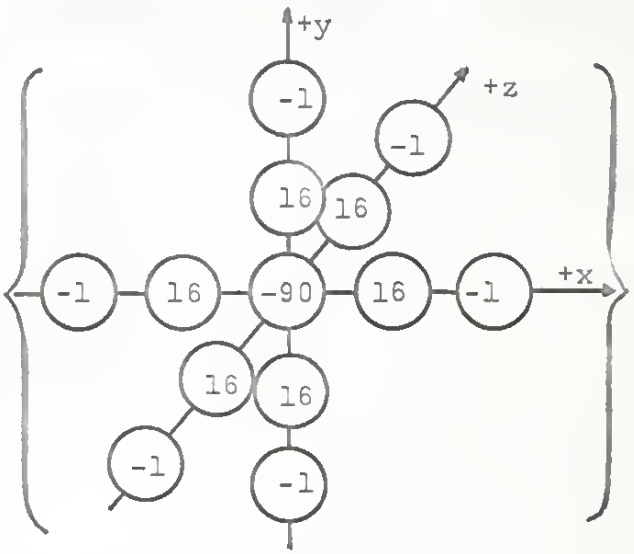
$$\left.\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}\right|_{x_0,y_0,z_0} = \nabla'^2_h\phi_{000}$$

$$= \frac{1}{h^2}\left[\phi_{-100} - 2\phi_{000} + \phi_{100} - \frac{1}{12}(\phi_{-200} - 4\phi_{-100} + 6\phi_{000} - 4\phi_{100} + \phi_{200}) + \phi_{0-10} - 2\phi_{000} + \phi_{010} - \frac{1}{12}(\phi_{0-20} - 4\phi_{0-10} + 6\phi_{000} - 4\phi_{010} + \phi_{020}) + \phi_{00-1} - 2\phi_{000} + \phi_{001} - \frac{1}{12}(\phi_{00-2} - 4\phi_{00-1} + 6\phi_{000} - 4\phi_{001} + \phi_{002})\right]$$

$$= \frac{1}{12h^2}\left[-(\phi_{-200} + \phi_{0-20} + \phi_{00-2} + \phi_{200} + \phi_{020} + \phi_{002}) + 16(\phi_{-100} + \phi_{0-10} + \phi_{00-1} + \phi_{100} + \phi_{010} + \phi_{001}) - 90\phi_{000}\right]$$

In symbolic notation,

$$\nabla'^2_h\phi_{000} = \frac{1}{12h^2}$$



As mentioned previously, this approximation is very seldom used because of the additional values required at points adjacent to

but outside the boundary.

Now consider a simple problem similar to the one discussed for the two-dimensional case. Assume that the lattice for this example forms a cube extending a distance of 4h in each dimension. Hence, we have a lattice of 125 points including 27 interior points. These interior points can be numbered in turn beginning at the upper left hand corner of the first interior plane. Move to the right to the last interior point in this row. Then continue to the second row, the third and so on. When this plane is completed, continue to the second interior plane.

The lattice points on the exterior are also numbered beginning at the upper left. Number the first plane by rows moving from left to right. The planes containing interior points will be numbered clockwise, beginning in the upper left hand corner. The last plane will be numbered using the same pattern used on the first plane. See figure 4 for an example of the numbering of lattice points.

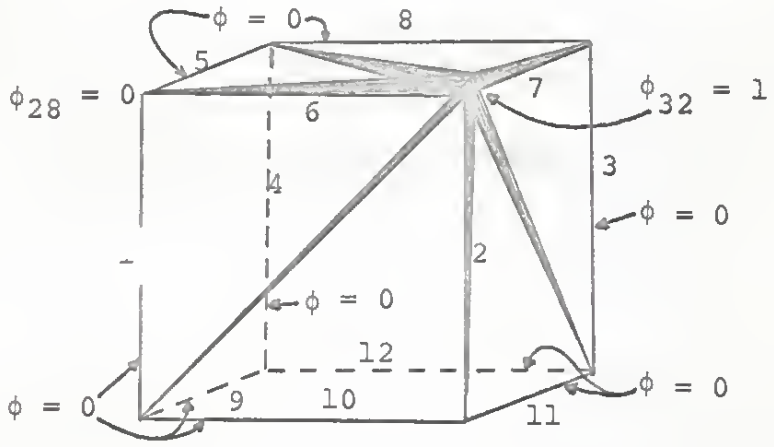


Fig. 3

In the future, lattice points will be referred to using single subscript notation. The single subscript will refer to the point designated by this subscript in figure 4.

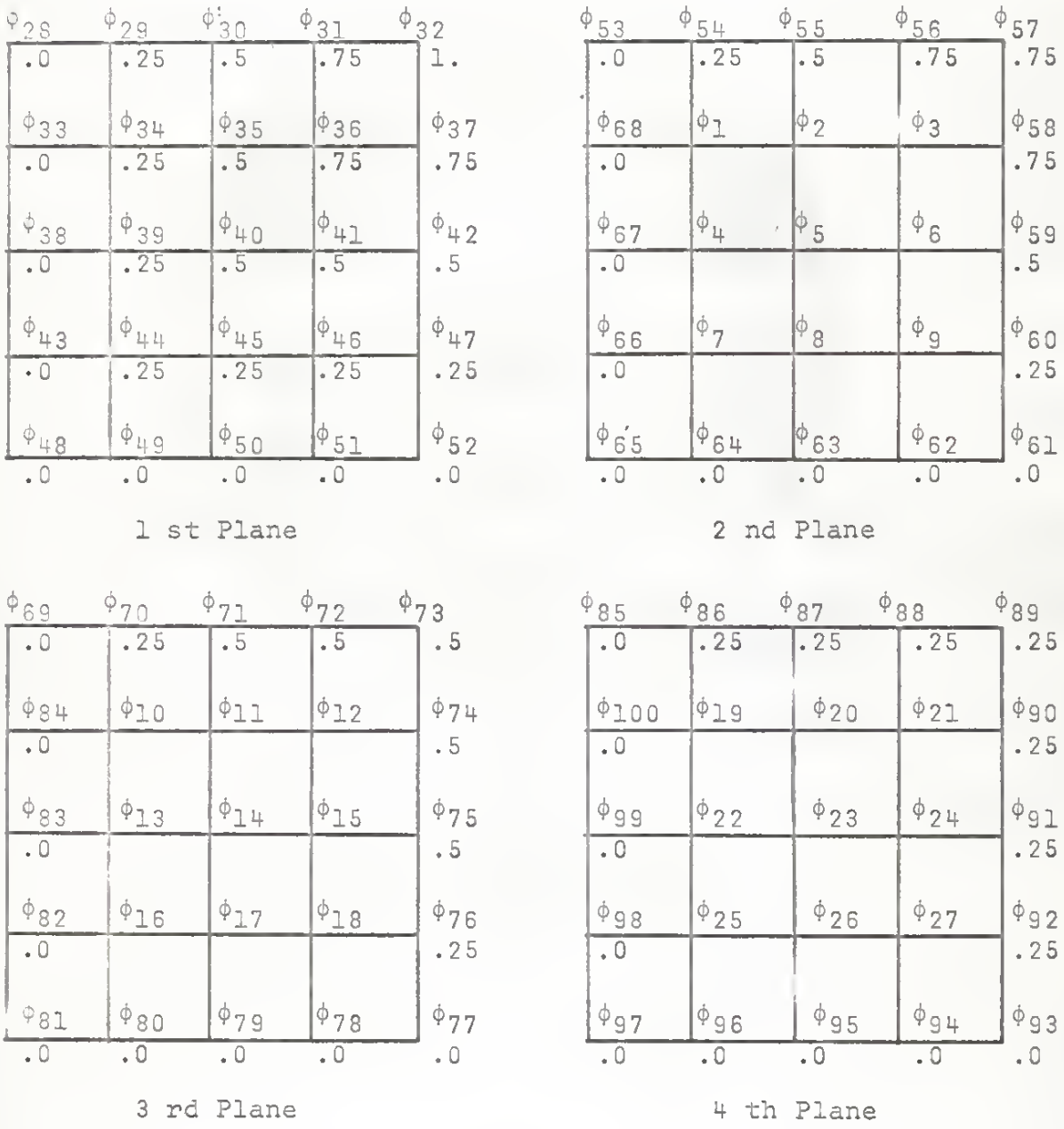


Fig. 4

Assume that we have a function defined on the boundary which gives a maximum potential of one at ϕ_{32} (see figure 3), and the values at other points of the boundary as given in figure 4. Label the edges of the region from 1 through 12. Let the potential at all points on edges 1, 3, 4, 5, 8, 9, 10, 11 and 12 be zero. Also, let all points on the three planes determined by these edges have a potential of zero. The potential at the remaining boundary points can be defined by imagining the potential as the thickness in the walls of the region. Note, at point 32, the thickness is one. The fifth plane is not shown as the potential at each point is zero. The lattice points are numbered following the same pattern used to number the first plane.

To demonstrate the procedure using the three-dimensional Laplacian, determine the first approximation to ϕ_1. Since the maximum is one and the minimum is zero on the boundary, let the initial guess again by 0.5 for all interior points. From the formula derived for the three-dimensional Laplacian, we need the values for six neighboring points. For ϕ_1 the six points are 34, 54 and 68 on the boundary and 2, 4 and 10 from the interior. Hence,

[6] $$\phi_1 = 1/6\ (\phi_2 + \phi_4 + \phi_{10} + \phi_{34} + \phi_{54} + \phi_{68})$$

$$= 1/6\ (.5 + .5 + .25 + .5 + .25 + 0) = 0.333$$

This same procedure will be used on all interior points, in order. For this problem, there will be a total of 27 equations, one for

each interior point. Equation 6 can be written in the form:

$$-6\phi_1 + \phi_2 + \phi_4 + \phi_{10} = -(\phi_{34} + \phi_{54} + \phi_{68}).$$

Consider the matrix of the coefficients of the system of equations. Again, the matrix is strongly diagonal. The absolute value of each diagonal element is at least as large as the sum of the remaining elements in that row or column.

When the values at all interior points have been corrected, the process begins over again at ϕ_1. When the value of the potential at all interior points converges, to a specified accuracy, the solution has been obtained. It is obvious that by hand, this method would prove to be very tedious and slow, even for this relatively "small" problem. However, the method is easily adapted to a computer and the FORTRAN program along with the results are given in appendix B. Note, that only the second, third and fourth planes of lattice points are listed, as the first and fifth consist only of boundary points with constant values. Observe also that the values at the boundary points of the planes illustrated remain constant.

EXTENSION OF GAUSS-SEIDEL METHOD

The Liebmann improvement matrix technique is nothing more than the Gauss-Seidel method applied to a particular problem. The first extension of the Gauss-Seidel takes a set of equations with a strongly diagonal matrix, and uses the latest approximations for all variables. It can also be done using matrices, but it does require inversion of a triangular matrix.

To illustrate the method, refer to the two-dimensional problem discussed on page 5. From the first two difference equations for the system, we can obtain the following by rearranging and solving for ϕ_j, where j is the index of the lattice point:

$$\phi_1 = 1/4\ (\phi_2 + \phi_4 + \phi_{11} + \phi_{25})$$

$$\phi_2 = 1/4\ (\phi_1 + \phi_3 + \phi_5 + \phi_{12})\ .$$

Remember that ϕ_j, $j = 10, 11, \ldots, 25$, are boundary points with constant values.

Consider the equation for the potential ϕ_2, which is a linear combination of ϕ_1, ϕ_3, ϕ_5 and ϕ_{12}. Note that ϕ_1 appears in the expression for ϕ_2. The newest approximation for ϕ_1 could be used. Similarly, by the time we determine the potential at the third interior point, we have a new value for ϕ_2. The value of the potential at all interior points, except the first, can be expressed in terms of some already corrected potential values.

In this example, use of the expression ϕ_s^{i+1} will represent the newest approximation for the potential of the internal

lattice points, s = 1, 2, ... , 9. The approximation immediately preceding ϕ_s^{i+1} will be represented by ϕ_s^i.

With this notation in mind, we can now rewrite the nine equations for this example as:

$$\phi_1^{i+1} = \frac{1}{4}(\phi_2^i + \phi_4^i + \phi_{11} + \phi_{25})$$

$$\phi_2^{i+1} = \frac{1}{4}(\phi_1^{i+1} + \phi_3^i + \phi_5^i + \phi_{12})$$

$$\phi_3^{i+1} = \frac{1}{4}(\phi_2^{i+1} + \phi_6^i + \phi_{13} + \phi_{15})$$

$$\phi_4^{i+1} = \frac{1}{4}(\phi_1^{i+1} + \phi_5^i + \phi_7^i + \phi_{24})$$

$$\phi_5^{i+1} = \frac{1}{4}(\phi_2^{i+1} + \phi_4^{i+1} + \phi_6^i + \phi_8^i)$$

$$\phi_6^{i+1} = \frac{1}{4}(\phi_3^{i+1} + \phi_5^{i+1} + \phi_9^i + \phi_{16})$$

$$\phi_7^{i+1} = \frac{1}{4}(\phi_4^{i+1} + \phi_8^i + \phi_{21} + \phi_{23})$$

$$\phi_8^{i+1} = \frac{1}{4}(\phi_5^{i+1} + \phi_7^{i+1} + \phi_9^i + \phi_{20})$$

$$\phi_9^{i+1} = \frac{1}{4}(\phi_6^{i+1} + \phi_8^{i+1} + \phi_{17} + \phi_{19})$$

Expressed in matrix form, these equations can be written as

$$\phi_s^{(i+1)} = A\phi_s^{(i)} + B\phi_s^{(i+1)} + H$$

where H is a constant vector derived from the values of the potential at the boundary points. The method is similar to that used to obtain the matrix coefficients A and B. These matrices are illustrated on the following two pages.

$$
\begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \\ \phi_6 \\ \phi_7 \\ \phi_8 \\ \phi_9 \end{bmatrix}^{(i+1)} = \underbrace{\begin{bmatrix} 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{(A)} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \\ \phi_6 \\ \phi_7 \\ \phi_8 \\ \phi_9 \end{bmatrix}^{(i)} + \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 \end{bmatrix}}_{(B)} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \\ \phi_6 \\ \phi_7 \\ \phi_8 \\ \phi_9 \end{bmatrix}^{(i+1)} + H
$$

$$
\begin{bmatrix}
0 & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} \\
0 & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
.5 \\ 1.0 \\ .5 \\ .0 \\ .0 \\ .0 \\ .0 \\ .0 \\ .0 \\ .0 \\ .0 \\ .0 \\ .0 \\ .0 \\ .0 \\ .0
\end{bmatrix}
=
\begin{bmatrix}
.125 \\ .250 \\ .125 \\ .000 \\ .000 \\ .000 \\ .000 \\ .000 \\ .000
\end{bmatrix}
$$

This product is evaluated by using the boundary values for φ_j, $j = 10, 11, \ldots, 25$. By assigning letters to these matrices, we can simplify the appearance of the expression on the previous page.

$$I\phi_s^{(i+1)} = \phi_s^{(i+1)} = A\phi_s^{(i)} + B\phi_s^{(i+1)} + H$$

$$(I - B)\phi_s^{(i+1)} = A\phi_s^{(i)} + H$$

$$\phi_s^{(i+1)} = (I - B)^{-1}A\phi_s^{(i)} + (I - B)^{-1}H$$

$$\phi_s^{(i+1)} = T\phi_s^{(i)} + G$$

This matrix T is an "improvement" matrix for the Gauss-Seidel process. It is necessary, first, to calculate T and G.

After carrying out the calculations indicated above, the improvement matrix T and the new constant vector G appear as shown below. Again we can choose the initial approximation for the interior points as 0.5. For this example, then

$$\phi^{(i+1)} = \begin{bmatrix} 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{16} & \frac{1}{4} & \frac{1}{16} & \frac{1}{4} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{64} & \frac{1}{16} & \frac{1}{64} & \frac{1}{16} & \frac{1}{4} & 0 & 0 & 0 \\ 0 & \frac{1}{16} & 0 & \frac{1}{16} & \frac{1}{4} & 0 & \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{32} & \frac{1}{16} & \frac{1}{32} & \frac{1}{8} & \frac{1}{4} & \frac{1}{16} & \frac{1}{4} & 0 \\ 0 & \frac{3}{256} & \frac{1}{32} & \frac{3}{256} & \frac{3}{64} & \frac{1}{8} & \frac{1}{64} & \frac{1}{16} & \frac{1}{4} \\ 0 & \frac{1}{64} & 0 & \frac{1}{64} & \frac{1}{16} & 0 & \frac{1}{16} & \frac{1}{4} & 0 \\ 0 & \frac{3}{256} & \frac{1}{64} & \frac{3}{256} & \frac{3}{64} & \frac{1}{16} & \frac{1}{32} & \frac{1}{8} & \frac{1}{4} \\ 0 & \frac{3}{512} & \frac{3}{256} & \frac{3}{512} & \frac{3}{128} & \frac{3}{64} & \frac{3}{256} & \frac{3}{64} & \frac{1}{8} \end{bmatrix} \begin{bmatrix} .5 \\ .5 \\ .5 \\ .5 \\ .5 \\ .5 \\ .5 \\ .5 \\ .5 \end{bmatrix} + \begin{bmatrix} .125 \\ .281 \\ .196 \\ .031 \\ .079 \\ .068 \\ .008 \\ .022 \\ .023 \end{bmatrix} = \begin{bmatrix} .375 \\ .593 \\ .399 \\ .343 \\ .484 \\ .346 \\ .211 \\ .298 \\ .163 \end{bmatrix}$$

$$(T) \qquad (\phi_s^0) \qquad (G) \qquad (\phi_s^1)$$

Compare these values, using the Gauss-Seidel method, with the values shown for this example in appendix A, after the first iteration. Note, since the values using the improvement matrix are nearer to the solution, we would expect that this method would converge faster than does the ordinary Liebmann method. A FORTRAN program is illustrated in appendix C and the results are shown for the 1^{st}, 5^{th} and 13^{th} iterations.

EXTENSION OF THE GAUSS-SEIDEL METHOD - THREE DIMENSIONAL CASE

It is also possible to apply the improvement matrix to the three dimensional example discussed earlier. We will use the numbering system previously described on page 18. In the previous section, we considered a problem in two dimensions. It was a relatively simple matter to find a system of equations to express the potential at the interior points. We then looked at the coefficients in these equations and placed these values in the proper locations in one of the matrices A, B or H.

In three dimensions, the method is very similar; we usually have more interior points and larger matrices to work with. The major requirement is that we have a specified pattern for numbering the lattice points. We will again have a system of equations, one equation for each interior point. By analyzing the coefficients in these equations, we can again form the matrices A, B or H by placing the values of these coefficients in the proper locations.

The values at the newest iteration, the $(i + 1)^{st}$, can be expressed in terms of the values at the i^{th} iteration, and the matrices A, B and H can be defined as follows:

$$[7] \qquad I\phi_s^{(i+1)} = \phi_s^{(i+1)} = A\phi_s^{(i)} + B\phi_s^{(i+1)} + H.$$

As an example, consider the equation for the potential ϕ_4, which is $\phi_4 = 1/6\ (\phi_1 + \phi_5 + \phi_7 + \phi_{13} + \phi_{39} + \phi_{67})$.

The number of the lattice point considered determines the row of the matrix in which the coefficients are placed; i.e., for the

equation above, the 4 indicates which row the coefficients are placed in. For this equation, we are placing coefficients in the fourth row of the matrices A, B and H. Since there are six neighboring points for this equation, we will place six coefficients in the fourth rows of these matrices. The matrix in which each of these six coefficients is placed is determined by the nature of each of the six points. If they are boundary points, the coefficients are placed in the fourth row of H. If they are interior points, they have either been improved in this particular iteration or they have not. If they have been improved in this iteration, the index is less than the index of the point being improved; i.e., ϕ_1, ϕ_2, and ϕ_3 have already been improved when we reach ϕ_4. Thus, these coefficients will be placed in the fourth row of B. If they have not been improved in this iteration, their indices are greater than four. These coefficients, then, will be placed in the fourth row of A.

The columns in which these six coefficients are placed are determined by the indices of the six neighboring points. For example, the 13 of ϕ_{13} implies that the coefficient of ϕ_{13} in the expression for ϕ_4 will be placed in column 13. We already know that this coefficient is also in row 4 of matrix A by the above discussion. Note that the coefficients of the six neighboring points are one-sixth, for all equations.

Similarly, this method can be extended to four or more dimensions. The most important precaution is that the lattice points be numbered in order. Secondly, the value at each point must be improved, in order, before returning to the starting

point. When the values at all interior points have been evaluated in terms of the six surrounding neighbors, equation 7 can be simplified to

$$\phi_s^{(i+1)} = T\phi_s^{(i)} + G$$

where $T = (I - B)^{-1}A$ and $G = (I - B)^{-1}H$.

Substituting the values for the boundary points into the expression above and choosing 0.5 as the original guess for the values at the interior points, ϕ_s^1 $(i = 0)$ can be written as:

$\phi_s^{(i+1)}$			
	0.3334	= 0.3473	= 0.2663
	0.4723	0.4700	0.3311
	0.6205	0.5152	0.3078
	0.3473	0.3658	0.2721
	0.4700	0.4677	0.3452
	0.5152	0.4998	0.3172
	0.2663	0.2721	0.1741
	0.3311	0.3452	0.2275
	0.3078	0.3172	0.1853
	s = 1, ..., 9;	10, ..., 18;	19, ..., 27

FIRST ITERATION

These are the values obtained from the FORTRAN program in appendix D. The matrices A, B, $(I - B)^{-1}$, T and G are not shown since all but G have dimensions 27 x 27, and would take up unnecessary space. If, however, these matrices should be required, a statement in the proper place in the FORTRAN program would cause this information to be printed or punched.

EXTENSION OF THE GAUSS-SEIDEL METHOD - WITHOUT MATRICES

The two methods used up to this point are the Liebmann method and an extension of the Gauss-Seidel method. The Liebmann method averages the four neighboring values, without taking into consideration the newest values. The extension of the Gauss-Seidel method makes use of the system of equations for the value of the potential at the interior points. From these equations, we manipulate matrices and finally come up with an improvement matrix. Clearly, from the preceding work, the extension of the Gauss-Seidel method should converge much faster than does the Liebmann method.

Consider, again, the two-dimensional Liebmann method discussed earlier. Use the same numbering system and the same pattern for improving the potential. In the FORTRAN program for the Liebmann method for this example (appendix A), there is a statement analogous to

$$\phi_s^{(k+1)} = 1/4\ [\phi_{i-1,j}^{(k)} + \phi_{i,j-1}^{(k)} + \phi_{i,j+1}^{(k)} + \phi_{i+1,j}^{(k)}]$$

where s = 1, 2, ..., 9; i = row index; j = column index. This says simply that at each interior point the value for the potential of the $(1 + 1)^{st}$ iteration is equal to the average of the values of the four neighboring points at the i^{th} iteration.

Notice, that by altering this expression only slightly, we have another method for approximating the value of the potential. This method is to use the newest values as they are available; i.e., by the time we correct the value of the potential ϕ_7, we

already have new values for the potential at points 1 through 6. Hence, the expression would now appear as

$$\phi_s^{(k+1)} = 1/4\ [\phi_{i-1,j}^{(k+1)} + \phi_{i,j-1}^{(k+1)} + \phi_{i,j+1}^{(k)} + \phi_{i+1,j}^{(k)}].$$

As the procedure implies, this is the method of successive displacements, sometimes called the Gauss-Seidel method. A FORTRAN program with the values after the 1^{st}, 5^{th} and 13^{th} iterations appears in appendix E. Compare these values with those shown in appendix C for the method using the improvement matrix. Note, the respective values are exactly the same in like numbered iterations.

The significance of this observation is if matrix techniques are utilized, then the Gauss-Seidel method should be used for computer work. This method does not depend on the inversion of matrices. However, if matrices are not used, it is better to use the extension of the Gauss-Seidel because it requires less storage then the Gauss-Seidel and converges faster than does the Liebmann method.

UNEQUAL ARM METHOD

The methods described in the previous pages are all well adapted to calculating the value of the potential at the interior points of lattices in which all points lie exactly on the nodes of the lattice. However, the boundary points of the region under observation will, in general, fall between lattice points.

This presents a new problem. We may have several points that have one or more of their neighboring lattice points lying outside the boundary (i.e., 1, 3, 5, 7 and 8 in figure 5). The potential at the interior points must be expressed using boundary points. For example, ϕ_1 would be expressed in terms of ϕ_j, ϕ_k, ϕ_2 and ϕ_4.

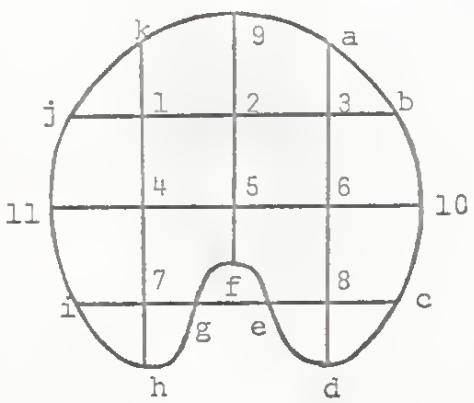


Fig. 5

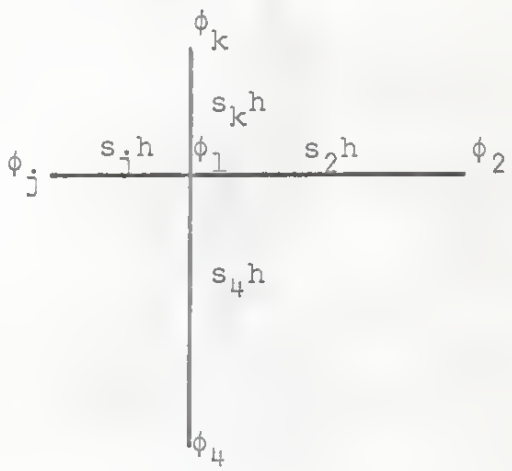


Fig. 6

Figure 6 illustrates the position of ϕ_1 and its neighbors. The concept which must be developed here relies on the different distances from ϕ_1 to each of its neighboring points.

Before a solution of this type of problem can be attempted, a method must be developed to express the potential at points for which the "arms" are unequal. To do this we must consider the first and second divided differences.[8]

$$1^{st}\ \text{D.D.}\quad f(x_0,x_1) = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{f(x_1)}{x_1 - x_0} - \frac{f(x_0)}{x_1 - x_0}$$

$$2^{nd}\ \text{D.D.}\quad f(x_0,x_1,x_2) = \frac{f(x_0,x_1) - f(x_1,x_2)}{x_0 - x_2}$$

$$= -\frac{f(x_1)}{(x_1 - x_0)(x_2 - x_1)} - \frac{f(x_0)}{(x_0 - x_2)(x_1 - x_0)} - \frac{f(x_2)}{(x_0 - x_2)(x_2 - x_1)}$$

We want a solution for Laplace's equation at all points in the region. Hence, we must have

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = \nabla^2\phi = 0.$$

If we approximate our function $\phi(x,0)$ by Newton's divided difference interpolating formula,[9] we can write

$$\phi(x,0) \simeq f(x,0) = f(x_0) + (x - x_0)f(x_0,x_1) + (x - x_0)(x - x_1)f(x_0,x_1,x_2)$$

Hence,

$$\frac{\partial^2\phi(x,0)}{\partial x^2} = 2f(x_0,x_1,x_2).$$

---

8. Kaiser S. Kunz, Numerical Analysis, McGraw-Hill Book Company, Inc., New York, London, Toronto, 1957, chapter 5.
9. Ibid.

In the definition for $f(x_0,x_1,x_2)$, let $x_1 = \phi_j$, $x_2 = \phi_2$ and $x_0 = \phi_1$. See figure 6 for an illustration of the points ϕ_i, $i = 1, 2, 4, j, k$. Note,

$$x_1 - x_0 = \phi_j - \phi_1 = - s_j h$$

$$x_2 - x_1 = \phi_2 - \phi_j = s_2 h - (-s_j h) = s_2 h + s_j h$$

$$x_0 - x_2 = \phi_1 - \phi_2 = - s_2 h$$

Then

$$\frac{\partial^2 \phi(x,0)}{\partial x^2} \approx 2\left\{\frac{-\phi_j}{-s_j h(s_2 h + s_j h)} - \frac{\phi_1}{-s_2 h(-s_j h)} - \frac{\phi_2}{-s_2 h(s_2 h + s_j h)}\right\}$$

$$= 1/h^2 \left\{\frac{2\,\phi_j}{s_j(s_2 + s_j)} - \frac{2\,\phi_1}{s_2 s_j} + \frac{2\,\phi_2}{s_2(s_2 + s_j)}\right\}.$$

Similarly,

$$\frac{\partial^2 \phi(0,y)}{\partial y^2} = 1/h^2 \left\{\frac{2\,\phi_4}{s_4(s_k + s_4)} - \frac{2\,\phi_1}{s_4 s_k} + \frac{2\,\phi_k}{s_k(s_k + s_4)}\right\}.$$

Hence,

[8] $$\nabla^2 \phi = 0 = 1/h^2 \left\{\frac{2\,\phi_4}{s_4(s_k + s_4)} + \frac{2\,\phi_j}{s_j(s_2 + s_j)} + \frac{2\,\phi_k}{s_k(s_k + s_4)} + \frac{2\,\phi_2}{s_2(s_2 + s_j)} - \left(\frac{2}{s_j s_2} + \frac{2}{s_k s_4}\right)\phi_1\right\}$$

For an example, consider a region defined by the function $f(x,y) = x^2/9 + y^2/4 - 1 = 0$. This is a very specialized sample problem; however, it makes use of the formulas developed above. In this case only two arms are shortened for ϕ_1. If we let $s_2 = s_4 = 1$; $s_j = s$; $s_k = t$ and consider figure 7, equation 8 reduces

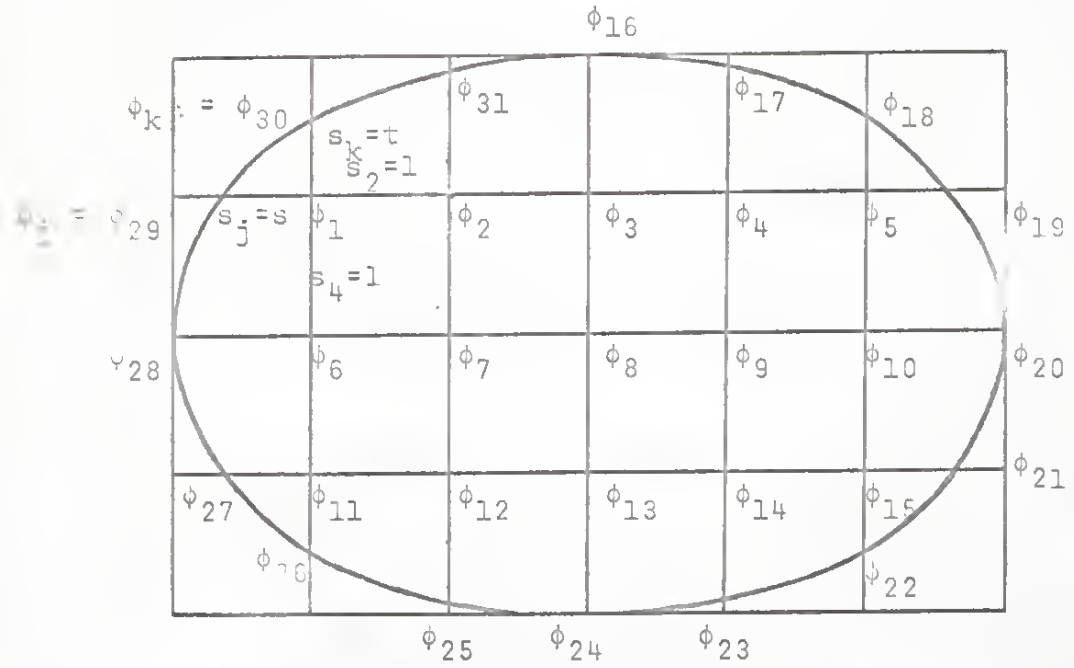


Fig. 7

to

[9] $$1/h^2 \left\{ \frac{2}{1+t}\phi_6 + \frac{2}{s(1+s)}\phi_j + \frac{2}{t(1+t)}\phi_k + \frac{2}{1+s}\phi_2 - \frac{2(s+t)}{st}\phi_1 \right\} = 0.$$

Since ϕ_j and ϕ_k are values at boundary points, they have constant values and equation 9 can be further simplified to

$$\phi_1 = A\phi_2 + B\phi_6 + C$$

where A, B and C are constants that are calculated once and for all for that particular "star".

Assume that a function has been defined so that the boundary points take on the values $\phi_{29} = \phi_{19} = 0.25$; $\phi_{30} = \phi_{18} = 0.50$; $\phi_{31} = \phi_{17} = 0.75$; $\phi_{16} = 1.0$ and all other 0.0.

Then the values of the unequal "arms" can be computed from the equation for the curve. For example, at point 29, f(x) = 1. When we determine the "x" we can compute the distance s. Using this method, then, we have s = 0.5980 and t = 0.4906. Equation 9 then becomes $\phi_1 = 0.1687\ \phi_2 + 0.1808\ \phi_6 + 0.2548$. Hence, if our initial guess for all interior points of the ellipse is 0.5, our first approximation for ϕ_1 is $\phi_1 = 0.1687\ (.5) + 0.1808\ (.5) + 0.2548 = 0.4296$.

From the symmetry of the boundary values for this example, we would expect that the potential at the interior points would also be symmetrical. That is, the potential would be the same at points symmetric to the y-axis. Table 15 of appendix F illustrates this.

Consider the potential ϕ_{14}. For this potential, figure 6 would become:

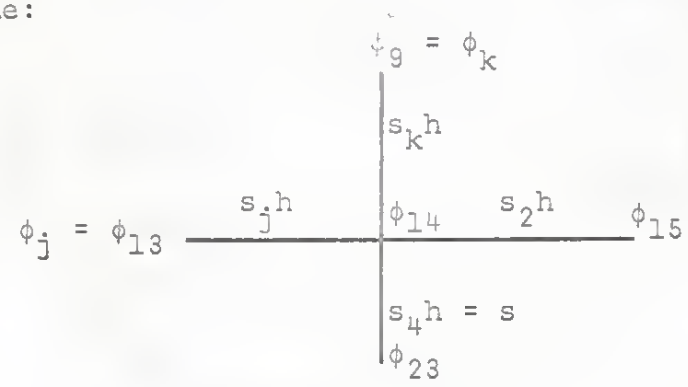


Fig. 8

Then $s_2 = s_k = s_j = 1$; $s_4 = s$. From equation 8, we have

$$\frac{1}{h^2}\left\{\frac{2\phi_{23}}{s(1+s)} + \frac{2\phi_{13}}{1(1+1)} + \frac{2\phi_9}{1(1+s)} + \frac{2\phi_{15}}{1(1+1)} - \left(\frac{2}{1\cdot 1} + \frac{2}{1\cdot s}\right)\phi_{14}\right\} = 0$$

This reduces to:

$$\frac{1}{h^2}\left\{\frac{2\phi_{23}}{s(1+s)} + \phi_{13} + \frac{2\phi_9}{1+s} + \phi_{15} - \frac{2s+2}{s}\phi_{14}\right\} = 0 \quad .$$

From the conditions on the boundary, we have that $\phi_{23} = 0$. Then the first term can be eliminated and we have

$$\phi_{13} + \frac{2}{1+s}\phi_9 + \phi_{15} - \frac{2s+2}{s}\phi_{14} = 0.$$

It follows that

$$\phi_{14} = \frac{s}{2s+2}\left\{\phi_{13} + \frac{2}{1+s}\phi_9 + \phi_{15}\right\}$$

which is in the form

$$\phi_{14} = A\phi_{13} + B\phi_9 + C\phi_{15} \quad .$$

From the equation $x^2/9 + y^2/4 - 1 = 0$, we can find the value of $y = f(x,y) = 1.8856$ at point 23. So $s = 1.8856 - 1.0 = 0.8856$. Hence,

$$\phi_{14} = 0.2348\phi_{13} + 0.2491\phi_9 + 0.2348\phi_{15} \quad .$$

The FORTRAN program for this example makes use of the newest values as they are available. Hence, for ϕ_{14}, we have already found new values for ϕ_9 and ϕ_{13}. Combining these values with the initial guess of 0.5 in the first iteration, we have

$$\phi_{14} = 0.2348\phi_{13} + 0.2491\phi_9 + 0.2348\ (0.5)$$

$$= 0.2348\phi_{13} + 0.2491\phi_9 + 0.1174$$

See appendix F for the FORTRAN program and potential values.

APPENDIX

APPENDIX A

```
C       PURPOSE
C          TO DETERMINE THE POTENTIAL FOR ALL INTERIOR POINTS
C           OF A TWO DIMENSIONAL LATTICE.
C
C       PARAMETERS
C          M  -  NUMBER OF POINTS IN THE X DIRECTION.
C          N  -  NUMBER OF POINTS IN THE Y DIRECTION
C          G  -  INITIAL GUESS FOR THE INTERIOR POINTS
C
C       REMARKS
C          IN THE DIMENSION STATEMENT, FIRST SUBSCRIPT IS A
C          COUNTER.  THIS COUNTS THE NUMBER OF ITERATIONS
C          REQUIRED TO OBTAIN THE ACCURACY DESIRED BY
C          STATEMENT 53.
C
C       METHOD
C          USES THE LIEBMANN METHOD
C
      DIMENSION PHI(30,5,5), XPHI(30,5,5)
    1 FORMAT (I2)
    2 FORMAT (F8.4)
    3 FORMAT (5F5.2)
   11 FORMAT (5F8.4//)
      I=0
      READ(1,1)M
      READ(1,1)N
      READ(1,2)G
      READ(1,3)((PHI(I,J,K),K=1,N),J=1,M)
      M1=M-I
      N1=N-I
      DO 21 J=2,M1
      DO 21 K=2,N1
   21 PHI(I,J,K)=G
   22 I=I+1
   23 IF(I.GE.21) GO TO 42
      DO 31 J=I,M
      DO 31 K=1,N
      IF(J.EQ.I)GO TO 51
      IF(K.EQ.I)GO TO 51
      IF(J.EQ.M)GO TO 51
      IF(K.EQ.N)GO TO 51
      PHI(I,J,K)=(PHI(I-1,J-I,K)+PHI(I-I,J,K+I)+PHI(I-I,J+1,K)+
     1PHI(I-1,J,K-1))/4.
   31 CONTINUE
   32 WRITE(3,11)((PHI(I,J,K),K=I,N),J=I,M)
      GO TO 52
   51 PHI(I,J,K)=PHI(I-I,J,K)
      GO TO 31
   52 DO 41 J=2,M1
      DO 41 K=2,N1
      XPHI(I,J,K)=ABS(PHI(I,J,K)-PHI(I-I,J,K))
   53 IF(XPHI(I,J,K).GE..0001)GO TO 22
   41 CONTINUE
```

```
42 STOP
   END
```

0.0000	0.5000	1.0000	0.5000	0.0000
0.0000	0.3750	0.6250	0.3750	0.0000
0.0000	0.3750	0.5000	0.3750	0.0000
0.0000	0.2500	0.3750	0.2500	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000

TABLE 1 --- 1 ST ITERATION

0.0000	0.5000	1.0000	0.5000	0.0000
0.0000	0.2712	0.4383	0.2712	0.0000
0.0000	0.1348	0.2031	0.1348	0.0000
0.0000	0.0569	0.0812	0.0569	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000

TABLE 2 --- 10 TH ITERATION

0.0000	0.5000	1.0000	0.5000	0.0000
0.0000	0.2636	0.4289	0.2636	0.0000
0.0000	0.1253	0.1880	0.1253	0.0000
0.0000	0.0494	0.0717	0.0494	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000

TABLE 3 --- 20 TH ITERATION

APPENDIX 3

```
C         PURPOSE
C            TO DETERMINE THE POTENTIAL FOR ALL INTERIOR POINTS
C            OF A THREE DIMENSIONAL LATTICE.
C
C         PARAMETERS
C            N  -  NUMBER OF POINTS IN THE Y DIRECTION
C            M  -  NUMBER OF POINTS IN THE X DIRECTION
C            MN -  NUMBER OF POINTS IN THE Z DIRECTION
C            G  -  INITIAL GUESS READ IN
C
C         REMARKS
C            IN THE DIMENSION STATEMENT, FIRST SUBSCRIPT IS A
C            COUNTER.  THIS KEEPS TRACK OF THE NUMBER OF
C            ITERATIONS REQUIRED TO OBTAIN THE ACCURACY
C            DESIRED BY STATEMENT 53.  THE REMAINING SUBSCRIPTS
C            ARE FOR THE COMMON X,Y,Z, DIRECTION IN SPACE.
C
C         METHOD
C            USES THE LIEBMANN METHOD
C
      DIMENSION PH1(30,5,5,5), XPH1(30,5,5,5)
    1 FORMAT(I2)
    2 FORMAT(F8.4)
    3 FORMAT(5F5.2)
   11 FORMAT(5F12.4//)
   14 FORMAT(1H1,I4,12HTH ITERATION//)
      I=0
      READ(1,1)M
      READ(1,1)N
      READ(1,1)MN
      READ(1,2)G
      READ(1,3)(((PHI(I,J,K,MK),K=1,N),J=1,M),MK=1,MN)
      I=0
      M1=M-1
      N1=N-1
      MN1=MN-1
      DO 21 J=2,M1
      DO 21 K=2,N1
      DO 21 MK=2,MN1
   21 PH1(1,J,K,MK)=G
   22 I=I+1
   23 IF(I.GE.21)GO TO 42
      DO 31 MK=1,MN
      DO 31 J=1,M
      DO 31 K=1,N
      IF(MK.EQ.1)GO TO 51
      IF(J.EQ.1)GO TO 51
      IF(K.EQ.1)GO TO 51
      IF(J.EQ.M)GOTO51
      IF(K.EQ.N)GO TO 51
      IF(MK.EQ.MN)GO TO 51
      PHI(I,J,K,MK)=(PH1(I-1,J-1,K,MK)+PHI(I-1,J+1,K,MK)+PHI(I-
     1-1,J,K-1,MK)+PH1(I-1,J,K+1,MK)+PH1(I-1,J,K,MK-1)+PHI(I-1,
```

```
  2J,K,MK+1))/6.
31 CONTINUE
   WRITE(3,14)I
32 WRITE(3,11)(((PHI(I,J,K,MK),K=1,N),J=I,M),MK=1,MN)
   GO TO 52
51 PHI(1,J,K,MK)=PHI(I-1,J,K,MK)
   GO TO 31
52 DO 41 J=2,MI
   DO 41 MK=2,MK1
   DO 41 K=2,N1
   XPHI(I,J,K,MK)=ABS(PHI(I,J,K,MK)-PHI(I-1,J,K,MK))
53 IF(XPHI(I,J,K,MK).GE..0001)GO TO 22
41 CONTINUE
42 STOP
   END
```

0.0000	0.2500	0.5000	0.7500	0.7500
0.0000	0.3333	0.5000	0.6250	0.7500
0.0000	0.3750	0.5000	0.5000	0.5000
0.0000	0.2917	0.3750	0.3333	0.2500
0.0000	0.0000	0.0000	0.0000	0.0000

2 ND PLANE

0.0000	0.2500	0.5000	0.5000	0.5000
0.0000	0.3750	0.5000	0.5000	0.5000
0.0000	0.4167	0.5000	0.5000	0.5000
0.0000	0.3333	0.4167	0.3750	0.2500
0.0000	0.0000	0.0000	0.0000	0.0000

3 RD PLANE

0.0000	0.2500	0.2500	0.2500	0.2500
0.0000	0.2917	0.3750	0.3333	0.2500
0.0000	0.3333	0.4167	0.3750	0.2500
0.0000	0.2500	0.3333	0.2917	0.2500
0.0000	0.0000	0.0000	0.0000	0.0000

4 TH PLANE

TABLE 4 --- 1 ST ITERATION

0.0000	0.2500	0.5000	0.7500	0.7500
0.0000	0.2037	0.3978	0.5740	0.7500
0.0000	0.1620	0.3045	0.3978	0.5000
0.0000	0.1055	0.1620	0.2037	0.2500
0.0000	0.0000	0.0000	0.0000	0.0000

2 ND PLANE

0.0000	0.2500	0.5000	0.5000	0.5000
0.0000	0.1620	0.3045	0.3978	0.5000
0.0000	0.1080	0.2064	0.3045	0.5000
0.0000	0.0586	0.1080	0.1620	0.2500
0.0000	0.0000	0.0000	0.0000	0.0000

3 RD PLANE

0.0000	0.2500	0.2500	0.2500	0.2500
0.0000	0.1055	0.1620	0.2037	0.2500
0.0000	0.0586	0.1080	0.1620	0.2500
0.0000	0.0293	0.0586	0.1055	0.2500
0.0000	0.0000	0.0000	0.0000	0.0000

4 TH PLANE

TABLE 5 --- 20 TH ITERATION

APPENDIX C

```
C         PURPOSE
C            TO DETERMINE THE POTENTIAL FOR ALL INTERIOR POINTS
C            OF A TWO DIMENSIONAL LATTICE.
C
C         PARAMETERS
C            M  -  NUMBER OF POINTS IN THE X DIRECTION
C            N  -  NUMBER OF POINTS IN THE Y DIRECTION
C            GU -  INITIAL GUESS
C
C         REMARKS
C            SUBROUTINE TO INVERT MATRIX REQUIRED.
C
C         METHOD
C            USES THE LIEBMANN IMPROVEMENT MATRIX.
C
      DIMENSION TN(9),XPHI(20,9),PPHI(20,9)
      DIMENSION XID(9,9),BIB(10,9,9),BIBI(I,9,9),T(9,9),G(9)
      DIMENSION PHI(25),A(9,9),B(9,9),H(9,25),ALAT(5,5),HP(9)
    1 FORMAT(I2)
    2 FORMAT(F8.4)
    3 FORMAT(16F5.2/9F5.2)
      FORMAT(5F5.2)
   19 FORMAT(F16.4)
  211 FORMAT(5X,I4,I2HTH ITERATION/)
      READ(1,1)M
      READ(1,1)N
      READ(1,2)GU
      MN=M*N
C        BOUNDARY POINTS
      READ(1,3)(PHI(JA),JA=I,MN)
      READ(I,4)((ALAT(I,J),J=1,N),I=I,M)
      NN=N*N
      JMAX=(M-2)*(N-2)
      DO 22 I=1,JMAX
      DO 22 J=I,JMAX
      A(I,J)=0.
   22 B(I,J)=0.
      MNAX=MN-JMAX
      DO 23 I=1,JMAX
      DO 23 J=1,MNAX
   23 H(I,J)=0.
      M1=M-1
      N1=N-1
      MM=1
      DO 61 I=2,M1
      DO 61 J=2,N1
      PHI(MM)=(ALAT(I-1,J)+ALAT(I,J-1)+ALAT(I,J+1)+ALAT(I+1,J)
     1)/4.
      IF((I-1).EQ.I)GO TO 62
      I1=MM-(N-2)
      B(MM,I1)=.25
   65 IF((J-1).EQ.1)GO TO 63
      I2=MM-1
```

```
      B(MM,I2)=.25
   66 IF((J+1).EQ.N)GO TO 64
      I3=MM+1
      A(MM,I3)=.25
   67 IF((I+1).EQ.M)GO TO 69
      I4=MM+N-2
      A(MM,I4)=.25
      GO TO 61
   62 I1=JMAX+J
      H(MM,I1)=.25
      GO TO 65
   63 I2=NN-I+2
      H(MM,I2)=.25
      GO TO 66
   64 I3=JMAX+N+I-1
      H(MM,I3)=.25
      GO TO 67
   69 I4=JMAX+M+2*N-J-1
      H(MM,I4)=.25
   61 MM=MM+1
      JM1=JMAX+1
      J=1
      DO 101 I=1,JMAX
      HP(I)=0.
      DO 101 K=JM1,MN
      HP(I)=HP(I)+H(I,K)*PHI(K)
  101 CONTINUE
      DO 111 I=1,JMAX
      DO 111 J=1,JMAX
      IF(I.EQ.J) GO TO 102
      XIO(I,J)=0.
      GO TO 111
  102 XIO(I,J)=1.
  111 CONTINUE
C        SUBTRACT B FROM I
      DO 121 J=1,JMAX
      DO 121 K=1,JMAX
  121 BIB(1,J,K)=XIO(J,K)-B(J,K)
C
      CALL MINV(BIO,JMAX,BIBI)
C
C        MULTIPLY I-B INVERSE AND A
      DO 131 I=1,JMAX
      DO 131 J=1,JMAX
      T(I,J)=0.
      DO 131 K=1,JMAX
      T(I,J)=T(I,J)+BIBI(1,I,K)*A(K,J)
  131 CONTINUE
      DO 141 I=1,JMAX
      G(I)=0.
      DO 141 K=1,JMAX
      G(I)=G(I)+BIBI(1,I,K)*HP(K)
  141 CONTINUE
```

```
      IC=1
      DO 151 I=1,JMAX
  151 PPHI(IC,I)=GU
  152 DO 171 I=I,JMAX
      TN(I)=0.
      DO 171 K=1,JMAX
      TN(I)=TN(I)+T(I,K)*PPHI(IC,K)
  171 CONTINUE
      IC=IC+1
      DO 191 I=1,JMAX
  191 PPHI(IC,I)=TN(I)+G(I)
      IIC=IC-1
      WRITE(3,211)IIC
      WRITE(3,19)(PPHI(IC,I),I=1,JMAX)
      DO 192 I=I,JMAX
      XPHI(IC,I)=ABS(PPHI(IC,I)-PPHI(IC-1,I))
      IF(XPHI(IC,I).GE..0001)GO TO 152
  192 CONTINUE
      STOP
      END

      SUBROUTINE MINV(AI,NI,AIBI)
C        FINDING THE INVERSE OF AN NXN MATRIX USING THE L.F.
C        METHOD.
      DIMENSION AI(10,9,9),BI(9,9,9),CI(9,9,9),DI(9,9,9)
      DIMENSION AIBI(I,9,9),TRACE(10),DETA(10),DETAIN(10)
      DIMENSION BIBI(1,9,9),ADJ(9,9,9)
      DO 406 I=1,NI
      DO 36 J=I,NI
      DO 36 K=1,NI
      IF(J.EQ.K)GO TO 35
      DI(I,J,K)=0.
      GO TO 36
   35 DI(I,J,K)=1.
   36 CONTINUE
      S=I
      TRACE(I)=0.
      DO 106 J=I,NI
      DO 106 K=1,NI
      IF(J-K)106,50,106
   50 TRACE(I)=TRACE(I)+AI(I,J,K)
  106 CONTINUE
      TRACE(I)=TRACE(I)/S
      DO 206 J=1,NI
      DO 206 K=1,NI
      CI(I,J,K)=0.
      CI(I,J,K)=CI(I,J,K)+TRACE(I)*DI(I,J,K)
  206 CONTINUE
      DO 306 J=1,NI
      DO 306 K=1,NI
      BI(I,J,K)=0.
      BI(I,J,K)=BI(I,J,K)+AI(I,J,K)-CI(I,J,K)
  306 CONTINUE
```

```
      DO 356 J=1,NI
      DO 356 K=1,NI
      AI(I+1,J,K)=0.
      DO 356 KJ=1,N1
      AI(I+1,J,K)=AI(I+1,J,K)+AI(1,J,KJ)*BI(I,KJ,K)
  356 CONTINUE
  406 CONTINUE
      I=1
      DO 506 J=1,NI
      DO 506 K=1,NI
      ADJ(I,J,K)=0.
      ADJ(I,J,K)=ADJ(I,J,K)+((-1.)**(NI-1))*BI(NI-1,J,K)
  506 CONTINUE
      DETA(I)=((-1.)**(NI-1))*TRACE(NI)
      DETAIN(I)=1./DETA(I)
      DO 606 J=1,NI
      DO 606 K=1,NI
      AIBI(1,J,K)=0.
      AIBI(1,J,K)=AIBI(1,J,K)+DETAIN(I)*ADJ(I,J,K)
  606 CONTINUE
      RETURN
      END
```

PHI(1)	0.3750	0.2855	0.2635
PHI(2)	0.5938	0.4509	0.4287
PHI(3)	0.3984	0.2746	0.2634
PHI(4)	0.3438	0.1474	0.1251
PHI(5)	0.4844	0.2100	0.1876
PHI(6)	0.3457	0.1362	0.1250
PHI(7)	0.2109	0.0603	0.0492
PHI(8)	0.2988	0.0827	0.0715
PHI(9)	0.1611	0.0547	0.0491
	TABLE 6	TABLE 7	TABLE 8
ITERATION	1 ST	5 TH	13 TH

APPENDIX

```
C         PURPOSE
C            TO DETERMINE THE POTENTIAL FOR ALL INTERIOR POINTS
C            OF A THREE DIMENSIONAL LATTICE
C
C         PARAMETERS
C            M  -  NUMBER OF POINTS IN THE X DIRECTION
C            N  -  NUMBER OF POINTS IN THE Y DIRECTION
C            MN -  NUMBER OF POINTS IN THE Z DIRECTION
C            GU -  INITIAL GUESS
C
C         REMARKS
C            SUBROUTINE REQUIRED TO INVERT MATRIX
C
C         METHOD
C            USES THE LIEBMANN IMPROVEMENT MATRIX.
C
      DIMENSION TN(27),XPHI(20,27),PHI(125),XID(27,27)
      DIMENSION A(27,27),B(27,27),H(27,125),ALAT(5,5,5),HP(27)
      DIMENSION BID(27,27),T(27,27),G(27),PPHI(20,27)
      DIMENSION IP(27),V(27)
      COMMON BID,IP,V
    1 FORMAT(I2)
    2 FORMAT(F8.4)
    3 FORMAT(16F5.2)
    4 FORMAT(5F5.2)
   19 FORMAT(1X,F8.4)
  211 FORMAT(5X,I4,12HTH ITERATION/)
      READ(1,1)M
      READ(1,1)N
      READ(1,1)MN
      READ(1,2)GU
      MN2=M*N*MN
C        BOUNDARY POINTS
      READ(1,3)(PHI(JA),JA=1,MN2)
      READ(1,4)(((ALAT(I,J,K),J=1,N),I=1,M),K=1,MN)
      JMAX=(M-2)*(N-2)*(MN-2)
      DO 22 I=1,JMAX
      DO 22 J=I,JMAX
      A(I,J)=0.
   22 B(I,J)=0.
      DO 23 I=1,JMAX
      DO 23 J=I,MN2
   23 H(I,J)=0.
      M1=M-1
      N1=N-1
      MN1=MN-1
      MM=1
      DO 61 K=2,MN1
      DO 61 I=2,M1
      DO 61 J=2,N1
      PHI(MM)=(ALAT(I,J,K-1)+ALAT(I-1,J,K)+ALAT(I,J-1,K)+ALAT(I
     1,J+1,K)+ALAT(I+1,J,K)+ALAT(I,J,K+1))/6.
      IF((K-1).EQ.1)GO TO 72
```

```
      I1=MM-(N-2)*(M-2)
      A(MM,I1)=.1667
   63 IF((I-1).EQ.1)GO TO 73
      I2=MM-(N-2)
      B(MM,I2)=.1667
   64 IF((J-1).EQ.1)GO TO 74
      I3=MM-1
      B(MM,I3)=.1667
   65 IF((J+1).EQ.N)GO TO 75
      I4=MM+1
      A(MM,I4)=.1667
      IF((I+1).EQ.M)GO TO 76
      I5=MM+(N-2)
      A(MM,I5)=.1667
   67 IF((K+1).EQ.MN)GO TO 77
      I6=MM+(N-2)*(M-2)
      A(MM,I6)=.1667
      GO TO 61
   72 I1=JMAX+N+N*(I-2)+J
      H(MM,I1)=.1667
      GO TO 63
   73 I2=JMAX+M*N+(K-2)*(2*N+2*(M-2))+J
      H(MM,I2)=.1667
      GO TO 64
   74 I3=JMAX+M*N+(K-1)*(2*N+2*(M-2))-I+2
      H(MM,I3)=.1667
      GO TO 65
   75 I4=JMAX+M*N+(K-2)*(2*N+2*(M-2))+N+I-1
      H(MM,I4)=.1667
      GO TO 66
   76 I5=JMAX+M*N+(K-2)*(2*N+2*(M-2))+2*N+M-1-J
      H(MM,I5)=.1667
      GO TO 67
   77 I6=JMAX+M*N+(K-1)*(2*N+2*(M-2))+(I-1)*N+J-1
      H(MM,I6)=.1667
   61 MM=MM+1
      JM1=JMAX+1
      DO 101 I=1,JMAX
      HP(I)=0.
      DO 101 K=JM1,MN2
      HP(I)=HP(I)+H(I,K)*PHI(K)
  101 CONTINUE
      DO 111 I=1,JMAX
      DO 111 J=1,JMAX
      IF(I.EQ.J) GO TO 102
      XID(I,J)=0.
      GO TO 111
  102 XID(I,J)=1.
  111 CONTINUE
C        SUBTRACT B FROM I
      DO 121 J=1,JMAX
      DO 121 K=1,JMAX
  121 BIB(J,K)=XID(J,K)-B(J,K)
C
```

```
      CALL CRAM(JMAX,1)
      CALL INVERS(JMAX)
C
C        MULTIPLY I-B INVERSE AND A
      DO 131 I=1,JMAX
      DO 131 J=I,JMAX
      T(I,J)=0.
      DO 131 K=I,JMAX
      T(I,J)=T(I,J)+BIB(I,K)*A(K,J)
  131 CONTINUE
      DO 141 I=1,JMAX
      G(I)=0.
      DO 141 K=I,JMAX
      G(I)=G(I)+BIB(I,K)*HP(K)
  141 CONTINUE
      IC=1
      DO 151 I=1,JMAX
  151 PPHI(IC,I)=GU
  152 DO 171 I=1,JMAX
      TN(I)=0.
      DO 171 K=I,JMAX
      TN(I)=TN(I)+T(I,K)*PPHI(IC,K)
  171 CONTINUE
      IC=IC+1
      DO 191 I=1,JMAX
  191 PPHI(IC,I)=TN(I)+G(I)
      IIC=IC-1
      WRITE(3,211)IIC
      WRITE(3,19)(PPHI(IC,I),I=1,JMAX)
      DO 192 I=1,JMAX
      XPHI(IC,I)=ABS(PPHI(IC,I)-PPHI(IC-1,I))
      IF(XPHI(IC,I).GE..0001)GO TO 152
  192 CONTINUE
      STOP
      END

      SUBROUTINE CRAM(N,I)
C     CROUT REDUCTION OF AUGMENTED MATRICES
C     THIS PROGRAM PERFORMS A CROUT REDUCTION ON A MATRIX AN.
C     WITH I=1. THE CROUT REDUCTION IS PERFORMED WITH ROW
C     INTERCHANGES.   WITH I=2. THE CROUT REDUCTION IS
C     PERFORMED WITHOUT ROW CHANGES.
      DIMENSION AN(27,27),IP(27),V(27)
      COMMON AN,IP,V
 2240 FORMAT(1X ,5HPIVOT,I3,19HIS LESS THAN 1.E-08)
 2241 FORMAT(1X ,5HPIVOT,I3,7HIS ZERO)
      GO TO (2200,2201),I
 2200 IDMV=1
      GOTO2202
 2201 IDMV=2
C     REDUCTION OF MATRIX
 2202 DO 2204 IDK=1,N
      V(IDK)=ABS(AN(IDK,1))
```

```
      DO2204 IDI=2,N
      IF(V(IDK)-ABS(AN(IDK,IDI)))2203,2204,2204
 2203 V(IDK)=ABS(AN(IDK,IDI))
 2204 CONTINUE
      DO 2222 IDK=1,N
      DETR=-1.
      IDK1=IDK-1
      DO2214 IDI=IDK,N
      DETPR=0.0
      IF(IDK-1)2208,2208,2206
 2206 DO2207 IDJ=1,IDK1
 2207 DETPR=DETPR+AN(IDI,IDJ)*AN(IDJ,IDK)
 2208 AN(IDI,IDK)=AN(IDI,IDK)-DETPR
      GO TO(2212,2225),IDMV
 2212 DETS=ABS(AN(IDI,IDK))/V(IDI)
      IF(DETS-DETR)2214,2214,2213
 2213 DETR=DETS
      IP(IDK)=IDK-IDI
      GO TO 2214
 2225 IP(IDK)=0
 2214 CONTINUE
      IDK2=IDK-IP(IDK)
      DETR=AN(IDK2,IDK)/V(IDK2)
      IF(ABS(DETR)-1.E-08)2230,2230,2232
 2230 WRITE(3,2240)IDK
      IF(AN(IDK2,IDK))2232,2231,2232
 2231 WRITE(3,2241)IDK
      CALL EXIT
 2232 V(IDK2)=V(IDK)
      V(IDK)=DETR
      DO2222 IDJ=1,N
      DETR=AN(IDK,IDJ)
      IF(IDJ-IDK)2215,2215,2216
 2215 AN(IDK,IDJ)=AN(IDK2,IDJ)
      GO TO 2220
 2216 DETPR=0.0
      IF(IDK-1)2219,2219,2217
 2217 DO2218 IDI=1,IDK1
 2218 DETPR=DETPR+AN(IDK,IDI)*AN(IDI,IDJ)
 2219 AN(IDK,IDJ)=(AN(IDK2,IDJ)-DETPR)/AN(IDK,IDK)
 2220 IF(IP(IDK))2221,2222,2222
 2221 AN(IDK2,IDJ)=DETR
 2222 CONTINUE
      RETURN
      END

      SUBROUTINE INVERS (N)
C     AFTER CALLING THE CRAM SUBROUTINE THIS SUBROUTINE WILL
C     COMPUTE THE INVERSE OF THE MATRIX AN AND STORE THE
C     INVERSE OF AN IN AN.
      DIMENSION AN(27,27),IP(27),V(27)
      COMMON AN,IP,V
      DO 2280 IDK2=1,N
```

```
      IDK = N+1-IDK2
      IDKI = IDK+1
      DO 2272 IDJ = IDKI,N
      V(IDJ)=AN(IDK,IDJ)
2272  AN(IDK,IDJ)=0.0
      AN(IDK,IDK)=1./AN(IDK,IDK)
      IF(IDK-1) 2276, 2276, 2273
2273  IDK3 = IDK-1
      DO 2275 IDJ2 = I, IDK3
      IDJ = IDK3+1-IDJ2
      DETPR = 0.0
      IDJ1 = IDJ+1
      IF(N-IDJ) 2275, 2275, 2279
2279  DO 2274 IDI = IDJI,IDK
2274  DETPR=DETPR+AN(IDK,IDI)*AN(IDI,IDJ)
2275  AN(IDK,IDJ)=-DETPR/AN(IDJ,IDJ)
2276  DO 2278 IDJ=1,N
      DETPR = 0.0
      IF(N-IDK) 2278, 2278, 2284
2284  DO 2277 IDI = IDKI,N
2277  DETPR=DETPR+V(IDI)*AN(IDI,IDJ)
2278  AN(IDK,IDJ)=AN(IDK,IDJ)-DETPR
2280  CONTINUE
      DO 2283 IDK2=I,N
      IDK = N+1-IDK2
      IDKP = IDK - IP(IDK)
      IF(IP(IDK)) 2281, 2283, 2283
2281  DO 2282 IDI = I,N
      DETR=AN(IDI,IDK)
      AN(IDI,IDK)=AN(IDI,IDKP)
2282  AN(IDI,IDKP)=DETR
2283  CONTINUE
      RETURN
      END
```

PHI(1)	0.2037	PHI(10)	0.1620	PHI(19)	0.1054
PHI(2)	0.3979	PHI(11)	0.3044	PHI(20)	0.1619
PHI(3)	0.5741	PHI(12)	0.3978	PHI(21)	0.2037
PHI(4)	0.1620	PHI(13)	0.1079	PHI(22)	0.0584
PHI(5)	0.3044	PHI(14)	0.2062	PHI(23)	0.1078
PHI(6)	0.3978	PHI(15)	0.3043	PHI(24)	0.1619
PHI(7)	0.1054	PHI(16)	0.0584	PHI(25)	0.0292
PHI(8)	0.1619	PHI(17)	0.1078	PHI(26)	0.0584
PHI(9)	0.2037	PHI(18)	0.1619	PHI(27)	0.1054

TABLE 9 --- 13 TH ITERATION

APPENDIX B

```
C        PURPOSE
C           TO DETERMINE THE POTENTIAL FOR ALL INTERIOR POINTS
C           OF A TWO DIMENSIONAL LATTICE.
C
C        PARAMETERS
C           M  -  NUMBER OF POINTS IN THE X DIRECTION.
C           N  -  NUMBER OF POINTS IN THE Y DIRECTION
C           G  -  INITIAL GUESS FOR THE INTERIOR POINTS
C
C        REMARKS
C           IN THE DIMENSION STATEMENT, FIRST SUBSCRIPT IS A
C           COUNTER.  THIS COUNTS THE NUMBER OF ITERATIONS
C           REQUIRED TO OBTAIN THE ACCURACY DESIRED BY
C           STATEMENT 53.
C
C        METHOD
C           USES THE EXTENSION OF THE GAUSS-SEIDEL METHOD
C
      DIMENSION PHI(21,5,5), XPHI(21,5,5)
    1 FORMAT (I2)
    2 FORMAT (F8.4)
    3 FORMAT (5F5.2)
   11 FORMAT (5F8.4//)
      I=0
      READ(1,1)M
      READ(1,1)N
      READ(1,2)G
      READ(1,3)((PHI(I,J,K),K=1,N),J=1,M)
      M1=M-1
      N1=N-1
      DO 21 J=2,M1
      DO 21 K=2,N1
   21 PHI(I,J,K)=G
   22 I=I+1
   23 IF(I.GE.21) GO TO 42
      DO 31 J=1,M
      DO 3  K=1,N
      IF(J.EQ.1)GO TO 51
      IF(K.EQ.1)GO TO 51
      IF(J.EQ.M)GO TO 51
      IF(K.EQ.N)GO TO 51
      PHI(I,J,K)=(PHI(I,J-1,K)+PHI(I-1,J,K+I)+PHI(I-I,J+1,K)+PH
     1I(I,J,K-1))/4.
   31 CONTINUE
   32 WRITE(3,11)((PHI(I,J,K),K=1,N),J=1,M)
      GO TO 52
   51 PHI(I,J,K)=PHI(I-1,J,K)
      GO TO 31
   52 DO 41 J=2,M1
      DO 41 K=2,N1
      XPHI(I,J,K)=ABS(PHI(I,J,K)-PHI(I-1,J,K))
   53 IF(XPHI(I,J,K).GE..0001)GO TO 22
   41 CONTINUE
```

```
42 STOP
   END
```

0.0000	0.0000	1.0000	0.5000	0.0000
0.0000	0.3750	0.5938	0.3984	0.0000
0.0000	0.3438	0.4844	0.3457	0.0000
0.0000	0.2109	0.2988	0.2111	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000

TABLE 10 --- 1 ST ITERATION

0.0000	0.5000	1.0000	0.5000	0.0000
0.0000	0.2855	0.4509	0.2746	0.0000
0.0000	0.1474	0.2100	0.1362	0.0000
0.0000	0.0603	0.0827	0.0547	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000

TABLE 11 --- 5 TH ITERATION

0.0000	0.5000	1.0000	0.5000	0.0000
0.0000	0.2635	0.4287	0.2634	0.0000
0.0000	0.1251	0.1876	0.1250	0.0000
0.0000	0.0492	0.0715	0.0491	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000

TABLE 12 --- 13 TH ITERATION

APPENDIX F

```
C       PURPOSE
C          TO DETERMINE THE POTENTIAL FOR ALL INTERIOR POINTS
C          OF A TWO DIMENSIONAL LATTICE.  THE BOUNDARY POINTS
C          OF THE LATTICE FORM AN ELLIPSE AND HENCE, ALL
C          POINTS ARE NOT EQUIDISTANT FROM ALL OTHER POINTS.
C
C       PARAMETERS
C          M  -  NUMBER OF POINTS IN THE X DIRECTION.
C          N  -  NUMBER OF POINTS IN THE Y DIRECTION.
C          G  -  INITIAL GUESS FOR THE INTERIOR POINTS.
C
C       REMARKS
C          S4,S3,S2 AND S1 REFER TO THE LENGTH OF THE ARMS
C          ASSOCIATED WITH EACH POINT.  THIS IS A PROGRAM
C          WRITTEN ESPECIALLY FOR AN ELLIPSE AND MAKES USE OF
C          THE SYMMETRIC PROPERTIES (I.E. - REGARDING THE
C          LENGTHS OF THE 'ARMS').
C
C       METHOD
C          USES THE EXTENSION OF THE GAUSS-SEIDEL AS APPLIED
C          TO THE PROBLEM OF UNEQUAL DISTANCES BETWEEN
C          POINTS (UNEQUAL ARM-STAR).
C
      DIMENSION PHI(20,5,7),XPHI(20,5,7),S4(5,7),S3(5,7)
      DIMENSION S2(5,7),S1(5,7),S(5,7)
    1 FORMAT(I2)
    2 FORMAT(F8.4)
    3 FORMAT(7F5.2)
   11 FORMAT(1H1,I4,12HTH ITERATION/)
   12 FORMAT(7F16.4//)
      READ(1,1)M
      READ(1,1)N
      READ(1,2)G
      READ(1,3)((PHI(I,J,K),K=1,N),J=I,M)
C        Y=SQRT(4.-4.*X*X/9.)
C        X=SQRT(9.-9.*Y*Y/4.)
      IF(M.LE.3)GO TO 101
      IF(N.LE.3)GO TO 101
C        ALL LOOPS THROUGH STATEMENT NUMBER 51 DETERMINE
C        THE VALUES OF S(I),I=1,2,3,4.
      W=M
      HM=W/2.+1.
      IM=HM
      U=N
      HN=U/2.+I.
      IN=HN
      N1=N-1
      MI=M-1
      IM1=IM-1
      IN1=IN-1
      DO 21 J=2,IM1
      DO 21 K=2,IN1
      KK=K-4
```

```
      K=KK
      Y=SQRT(4.-4.*X*X/9.)
      IY=Y
      YY=IY
      S2(J,K)=Y-YY
      JJ=J-3
      X=JJ
      Y=-Y
      X=SQRT(9.-9.*Y*Y/4.)
      IX=X
      XX=IX
      IF(K.GE.3)GO TO 22
      S3(J,K)=X-XX
      GO TO 23
   22 S3(J,K)=1.
   23 S4(J,K)=1.
   21 S1(J,K)=1.
      DO 31 K=IN,IN
      DO 31 J=2,IM1
      S4(J,K)=1.
      S3(J,K)=1.
      S2(J,K)=1.
      S1(J,K)=1.
   31 CONTINUE
      IN2=IN+1
      DO 41 J=2,IM1
      DO 41 K=IN2,N1
      K2=K-2*(K-4)
      S4(J,K)=S4(J,K2)
      S3(J,K)=S1(J,K2)
      S2(J,K)=S2(J,K2)
      S1(J,K)=S3(J,K2)
   41 CONTINUE
      DO 42 J=IM,IM
      DO 42 K=2,N1
      S4(J,K)=1.
      S3(J,K)=1.
      S2(J,K)=1.
   42 S1(J,K)=1.
      IM2=IM+1
      DO 51 J=IM2,M1
      DO 51 K=2,N1
      J1=J-IM
      J2=J-2*J1
      S4(J,K)=S2(J2,K)
      S3(J,K)=S3(J2,K)
      S2(J,K)=S4(J2,K)
      S1(J,K)=S1(J2,K)
   51 CONTINUE
      DO 61 J=2,M1
      DO 61 K=2,N1
   61 PHI(I,J,K)=G
      I=0
```

```
 71 I=I+1
    DO 8I J=1,M
    DO 8I K=1,N
    IF(J.EQ.1)GO TO 74
    IF(K.EQ.1)GO TO 74
    IF(J.EQ.M)GO TO 74
    IF(K.EQ.N)GO TO 74
    S(J,K)=S4(J,K)*S2(J,K)+S3(J,K)*S1(J,K)
    PHI(I,J,K)=S1(J,K)*S2(J,K)*S3(J,K)*PHI(I,J-1,K)/((S2(J,K)
   1+S4(J,K))*S(J,K))+SI(J,K)*S2(J,K)*S4(J,K)*PHI(I,J,K-1)/((
   2S1(J,K)+S3(J,K))*S(J,K))+S1(J,K)*S3(J,K)*S4(J,K)*PHI(I-1,
   3J+1,K)/((S2(J,K)+S4(J,K))*S(J,K))+S2(J,K)*S3(J,K)*S4(J,K)
   4*PHI(I-1,J,K+1)/((SI(J,K)+S3(J,K))*S(J,K))
 81 CONTINUE
    WRITE(3,II)I
 82 WRITE(3,12)((PHI(I,J,K),K=1,N),J=1,M)
    GO TO 52
 74 PHI(I,J,K)=PHI(I-1,J,K)
    GO TO 81
 52 DO 91 J=2,M1
    DO 91 K=2,N1
    XPHI(I,J,K)=ABS(PHI(I,J,K)-PHI(I-1,J,K))
    IF(XPHI(I,J,K).GE..0001)GO TO 71
 91 CONTINUE
101 CONTINUE
    STOP
    END
```

	0.5000	0.7500	1.0000	0.7500	0.5000	
0.2500	0.4295	0.5457	0.6364	0.5943	0.4454	0.2500
0.0000	0.3574	0.4758	0.5280	0.5306	0.3690	0.0000
0.0000	0.2160	0.3020	0.3325	0.3447	0.1941	0.0000
	0.0000	0.0000	0.0000	0.0000	0.0000	

TABLE I3 --- I ST ITERATION

	0.5000	0.7500	1.0000	0.7500	0.5000	
0.2500	0.3038	0.4700	0.5649	0.4671	0.3004	0.2500
0.0000	0.1670	0.2725	0.3143	0.2693	0.1631	0.0000
0.0000	0.0841	0.1309	0.1441	0.1290	0.0819	0.0000
	0.0000	0.0000	0.0000	0.0000	0.0000	

TABLE 14 --- 9 TH ITERATION

	0.5000	0.7500	1.0000	0.7500	0.5000	
0.2500	0.2980	0.4619	0.5574	0.4618	0.2979	0.2500
0.0000	0.1605	0.2636	0.3060	0.2635	0.1604	0.0000
0.0000	0.0804	0.1257	0.1394	0.1257	0.0803	0.0000
	0.0000	0.0000	0.0000	0.0000	0.0000	

TABLE 15 --- 18 TH ITERATION

ACKNOWLEDGMENT

The author wishes to express sincere appreciation to Dr. W. E. Fuller of the Department of Mathematics for his patient supervision and valuable suggestions offered during the preparation of this report.

BIBLIOGRAPHY

I. G. Petrovsky;
Lectures on Partial Differential Equations, translated from Russian by A. Shenitzer, Interscience Publishers, Inc., New York, 1954, chapter 3.

Ivo Babuska; Milan Prager; Emil Vitasek;
Numerical Processes in Differential Equations, SNTL - Publishers of Technical Literature, Prague, translated by Milada Bourvkova, printed in Czechoslovakia for John Wiley & Sons, Ltd., London, Interscience Publishers, 1966, chap. 5.

H. F. Weinberger;
Partial Differential Equations, University of Minnesota, Blaisdell Publishing Company, New York, Toronto, London, 1965, pp. 48 - 52.

Kaiser S. Kunz
Numerical Analysis, McGraw-Hill Book Company, Inc., New York, Toronto, London, 1957, chapters 5, 13.

Tom M. Apostol;
Calculus, Blaisdell Publishing Company, New York, London, 1961, pp. 318 - 320.

SOLUTION OF PARTIAL DIFFERENTIAL EQUATIONS
LIEBMANN METHOD

by

LARRY WAYNE DAY

B.S., Jamestown College, 1964

AN ABSTRACT OF A MASTER'S REPORT

submitted in partial fulfillment of the

requirements for the degree

MASTER OF SCIENCE

Department of Mathematics

KANSAS STATE UNIVERSITY
Manhattan, Kansas

1968

The Liebmann method for solving partial differential equations makes use of a grid of points, a lattice. The potential at any interior point of the lattice is defined to be one-fourth of the sum of the potentials at the four closest neighboring points. It is essential that the potential be known or that it can be measured on the boundary of this lattice. For a simple two-dimensional problem, we approximate the value of the potential at some interior point. Then we continue to approximate its value at the remaining points, making use of any previously corrected values.

The method does not change significantly when applied to a three-dimensional lattice. We again require boundary conditions to be defined. Each point must be considered, in order, as we move through the lattice approximating the value of the potential. The only significant difference is that we now have six, instead of four, neighboring points which lie at a distance of h units. We must derive an expression for the Laplacian in three dimensions. This is similar to the two-dimensional Laplacian and can also be extended to n dimensions. We again correct the value of the potential at each grid point as we move through the lattice. We could possibly use the second approximations to the Laplacian. This approximation makes use of the eight closest neighboring points in the lattice for the two-dimensional case. We seldom use this approximation, however, because of the added complication of the additional points.

Rather than correct the potential at each interior point

individually, we may correct all points simultaneously. To do this we use matrix operations. This 'improvement' matrix is derived from a matrix containing certain coefficients. The coefficients are taken from the equations obtained when applying the Liebmann method to the value at each point. After repeated applications of the improvement matrix, the potential for all points will converge to within a specified accuracy. These values are solutions to the system of equations which approximates the partial differential equation.

If the region we are considering can not be defined so that all boundary points fall exactly on a lattice point, we need to develop a new method. This will enable us to approximate the potential at the interior points in terms of the four closest neighbors, whether or not they are lattice points. This method is known as the "unequal arm" method. As a result, the potential at any point can be expressed as a function of the potential at the four closest neighbors and also the distance between each neighbor and the point whose value is being corrected.